ASYMPTOTIC FIXED POINTS FOR NONLINEAR CONTRACTIONS

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Recently, W. A. Kirk proved an asymptotic fixed point theorem for nonlinear contractions by using ultrafilter methods. In this paper, we prove his theorem under weaker assumptions. Furthermore, our proof does not use ultrafilter methods.

1. Introduction

There are many papers in the literature that discuss the asymptotic fixed point theory, in which the existence of the fixed points is deduced from the assumption on the iterates of an operator (e.g., [1, 6] and the references therein). Recently, Kirk [5] studied an asymptotic fixed point theorem concerning nonlinear contractions. He proved the following theorem [5, Theorem 2.1] by appealing to ultrafilter methods.

Theorem 1.1. *Let (M,d) be a complete metric space. Let $T : M \to M$ be a continuous mapping such that*

$$d(T^n x, T^n y) \le \phi_n(d(x,y)) \tag{1.1}$$

for all $x, y \in M$, where $\phi_n : [0,\infty] \to [0,\infty]$ and $\lim_{n\to\infty} \phi_n = \phi$ uniformly on the range of d. Suppose that ϕ and all ϕ_n are continuous, and $\phi(t) < t$ for $t > 0$. If there exists $x_0 \in M$ which has a bounded orbit $O(x_0) = \{x_0, Tx_0, T^2x_0, \dots\}$, then T has a unique fixed point $x_* \in M$ such that $\lim_{n\to\infty} T^n x = x_*$ for all $x \in M$.

In this paper, we prove Theorem 1.1 under weaker assumptions without the use of ultrafilter methods.

2. Main results

We need the following recursive inequality (cf. [2, Lemmas 2.1 and 3.1], [3, Lemmas 2.1 and 2.4], and [4, Lemma 1]).

Lemma 2.1. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be upper semicontinuous, that is, $\limsup_{t\to t_0} \phi(t) \le \phi(t_0)$ for all $t_0 \in \mathbb{R}_+$, and $\phi(t) < t$ for $t > 0$. Suppose that there exist two sequences of nonnegative real*

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numbers $\{u_n\}$ *and* $\{\epsilon_n\}$ *such that*

$$u_{2n} \leq \phi(u_n) + \epsilon_n, \tag{2.1}$$

where $\epsilon_n \to 0$ *as* $n \to \infty$. *Then either* $\sup u_n = \infty$ *or* $\liminf u_n = 0$.

Proof. Suppose that $b = \sup\{u_n\} < \infty$. Assume that $\liminf u_n \neq 0$. Then there exist $m > 0$ and $N_1 > 0$ such that $u_n > m$ for all $n > N_1$.

Since ϕ is upper semicontinuous, $\phi(t)/t$ is upper semicontinuous on $[m,b]$ and so that $L_m = \max\{\phi(t)/t,\ t \in [m,b]\} < 1$ due to the facts that $\phi(t) < t$ for $t > 0$ and that $\phi(t)/t$ achieves its maximum on $[m,b]$.

Let $\epsilon > 0$. By (2.1), there exists $N_2 > N_1$ such that

$$u_{2n} \leq \phi(u_n) + \epsilon \leq L_m u_n + \epsilon \tag{2.2}$$

for all $n > N_2$. Note that the contraction mapping $f(x) = L_m x + \epsilon$ has a unique fixed point $\epsilon/(1 - L_m)$ and $\lim_{n\to\infty} f^n(x) = \epsilon/(1 - L_m)$ for any real number x. Now for any $n > N_2$,

$$u_{2^2 n} \leq \phi(u_{2n}) + \epsilon \leq L_m u_{2n} + \epsilon \leq L_m f(u_n) + \epsilon = f^2(u_n). \tag{2.3}$$

By induction, $u_{2^k n} \leq f^k(u_n)$ for all k, so that $m \leq f^k(u_n)$. Letting $k \to \infty$, $m \leq \epsilon/(1 - L_m)$. This is impossible since $\epsilon > 0$ can be arbitrarily chosen. □

Theorem 2.2. *Let* (M,d) *be a complete metric space. Let* $T : M \to M$ *be such that*

$$d(T^n x, T^n y) \leq \phi_n(d(x,y)) \tag{2.4}$$

for all $x, y \in M$, *where* $\phi_n : [0,\infty] \to [0,\infty]$ *and* $\lim_{n\to\infty} \phi_n = \phi$ *uniformly on any bounded interval* $[0,b]$. *Suppose that* ϕ *is upper semicontinuous and* $\phi(t) < t$ *for* $t > 0$. *Furthermore, suppose there exists a positive integer* n_* *such that* ϕ_{n_*} *is upper semicontinuous and* $\phi_{n_*}(0) = 0$. *If there exists* $x_0 \in M$ *which has a bounded orbit* $O(x_0) = \{x_0, Tx_0, T^2x_0, \dots\}$, *then* T *has a unique fixed point* $x_* \in M$ *such that* $\lim_{n\to\infty} T^n x = x_*$ *for all* $x \in M$.

Proof. First we establish the uniqueness of the fixed point. Assume that T has two different fixed points z_1 and z_2. Then $d(z_1,z_2) = d(T^n z_1, T^n z_2) \leq \phi_n(d(z_1,z_2))$. Letting $n \to \infty$, $d(z_1,z_2) \leq \phi(d(z_1,z_2)) < d(z_1,z_2)$. This is a contradiction.

Without loss of generality, we set $\phi_n(0) = 0$ and $\phi(0) = 0$. Let b be the diameter of $\operatorname{clos}\{O(x_0)\}$. For a given $x \in \operatorname{clos}\{O(x_0)\}$, denote $a_n = d(T^{n+1}x, T^n x)$. Then

$$\begin{aligned} a_{2n} &= d(T^{2n+1}x, T^{2n}x) \\ &\leq \phi_n(d(T^{n+1}x, T^n x)) \quad (\text{by } (2.4)) \\ &= \phi(a_n) + [\phi_n(a_n) - \phi(a_n)]. \end{aligned} \tag{2.5}$$

Let $\epsilon_n = \phi_n(a_n) - \phi(a_n)$. Since $\phi_n \to \phi$ uniformly on $[0,b]$, $\epsilon_n \to 0$. By Lemma 2.1, $\liminf_{n\to\infty} a_n = 0$.

Assume that $\lim_{n\to\infty} a_n = 0$ is not true. Since $\liminf_{n\to\infty} a_n = 0$, there exists $n_0 > 0$ such that $a_{n_0} < \limsup_{n\to\infty} a_n$. We choose a sequence $n_0 < n_1 < n_3 < \cdots$ with $a_{n_0} < a_{n_i}$ for all $i = 1,2,\ldots$ as can be done by choosing $\lim_{i\to\infty} a_{n_i} = \limsup_{n\to\infty} a_n$. Then,

$$\begin{aligned} a_{n_0} < a_{n_i} &= d\left(T^{n_i+1}x, T^{n_i}x\right) \\ &\le \phi_{n_i-n_0}\left(d\left(T^{n_0+1}x, T^{n_0}x\right)\right) \quad \text{(by (2.4))} \\ &= \phi\left(a_{n_0}\right) + \left[\phi_{n_i-n_0}\left(a_{n_0}\right) - \phi\left(a_{n_0}\right)\right]. \end{aligned} \tag{2.6}$$

Letting $i \to \infty$, we have $a_{n_0} \le \phi(a_{n_0}) < a_{n_0}$, which is a contradiction. We conclude that $\lim_{n\to\infty} a_n = 0$.

We next show that $\{T^n x\}$ is a Cauchy sequence. For if not, there exist p_k and q_k such that $p_k > q_k$ for each k, and

$$\lim_{k\to\infty} d\left(T^{p_k}x, T^{q_k}x\right) = \delta > 0. \tag{2.7}$$

Without loss of generality, assume that

$$d\left(T^{p_k}x, T^{q_k}x\right) > \frac{\delta}{2} \quad \forall k. \tag{2.8}$$

Since $s - \phi(s)$ is lower semicontinuous, there exists $\epsilon_0 > 0$ such that

$$s - \phi(s) > \epsilon_0 \quad \forall s \in \left[\frac{\delta}{2}, b\right] \tag{2.9}$$

due to the fact that $\phi(s) < s$ for $s > 0$ and $s - \phi(s)$ achieves its minimum on $[\delta/2, b]$.

Since $\lim_{n\to\infty} \phi_n = \phi$ uniformly on $[\delta/2, b]$, there exists m_0 such that $\phi_{m_0}(s) < \phi(s) + \epsilon_0 < s$ for all $s \in [\delta/2, b]$. Now

$$\begin{aligned} &d\left(T^{p_k}x, T^{q_k}x\right) \\ &\quad \le d\left(T^{p_k}x, T^{p_k+1}x\right) + d\left(T^{p_k+1}x, T^{p_k+2}x\right) + \cdots + d\left(T^{p_k+m_0-1}x, T^{p_k+m_0}x\right) \\ &\qquad + d\left(T^{p_k+m_0}x, T^{q_k+m_0}x\right) + d\left(T^{q_k+m_0}x, T^{q_k+m_0-1}x\right) + \cdots + d\left(T^{q_k+1}x, T^{q_k}x\right) \\ &\quad \le a_{p_k} + a_{p_k+1} + \cdots + a_{p_k+m_0-1} + \phi_{m_0}\left(d\left(T^{p_k}x, T^{q_k}x\right)\right) + a_{q_k+m_0-1} + \cdots + a_{q_k} \\ &\quad < a_{p_k} + a_{p_k+1} + \cdots + a_{p_k+m_0-1} + \phi\left(d\left(T^{p_k}x, T^{q_k}x\right)\right) + \epsilon_0 + a_{q_k+m_0-1} + \cdots + a_{q_k}. \end{aligned} \tag{2.10}$$

Letting $k \to \infty$ and using (2.7), $\lim_{n\to\infty} a_n = 0$, and the upper semicontinuity of ϕ,

$$\delta \le \phi(\delta) + \epsilon_0 < \delta. \tag{2.11}$$

This is a contradiction. Hence $\{T^n x\}$ is a Cauchy sequence, there exists $x_* \in M$ such that $\lim_{n\to\infty} T^n x = x_*$.

Now for each $n > 0$, $d(T^{n_*+n}x, T^{n_*}x_*) \le \phi_{n_*}(d(T^n x, x_*))$. Since $\limsup_{n\to\infty} \phi_{n_*}(d(T^n x, x_*)) \le \phi_{n_*}(0) = 0$, we have $\lim_{n\to\infty} T^n x = T^{n_*}x_*$, so that $T^{n_*}x_* = x_*$. Note that $T^{n_*}(Tx_*) = T(T^{n_*}x_*) = Tx_*$. By the uniqueness of the fixed point of T^{n_*}, $Tx_* = x_*$.

For any $y_0 \in M \setminus \{x_0\}$,

$$d(T^n y_0, T^n x_0) \le \phi_n(d(x_0, y_0)) \longrightarrow \phi(d(x_0, y_0)) < d(x_0, y_0) \tag{2.12}$$

as $n \to \infty$. Hence, $O(y_0)$ is also bounded. By the previous argument, $\lim_{n\to\infty} T^n y_0 = x_*$ due to the uniqueness of the fixed point. □

Remark 2.3. Kirk's paper [5] assumes the continuity for ϕ and all ϕ_n. We only assume the upper semicontinuity of ϕ and one of the ϕ_n's, which is weaker and easier to check.

If we have $\limsup_{t\to\infty}(\phi(t)/t) < 1$, then the assumption of the existence of a bounded orbit in Theorem 2.2 can be removed. This observation is formulated as the following corollary.

Corollary 2.4. *Let (M,d) be a complete metric space. Let $T : M \to M$ be such that*

$$d(T^n x, T^n y) \le \phi_n(d(x,y)) \tag{2.13}$$

for all $x, y \in M$, where $\phi_n : [0,\infty] \to [0,\infty]$ and $\lim_{n\to\infty} \phi_n = \phi$ uniformly on any bounded interval $[0,b]$. Suppose that ϕ is upper semicontinuous, $\phi(t) < t$ for $t > 0$, and $\limsup_{t\to\infty}(\phi(t)/t) < 1$. If there exists a positive integer n_* such that ϕ_{n_*} is upper semicontinuous and $\phi_{n_*}(0) = 0$, then T has a unique fixed point $x_* \in M$ such that $\lim_{n\to\infty} T^n x = x_*$ for all $x \in M$.

Proof. Examining the proofs of Lemma 2.1 and Theorem 2.2, one can find that the boundedness of the orbit $O(x)$ is only used to guarantee that

$$\begin{gathered} \sup\left\{\frac{\phi(t)}{t} : t \in [m,b]\right\} < 1, \\ \inf\{t - \phi(t) : t \in [m,b]\} > 0 \end{gathered} \tag{2.14}$$

for some $b > 0$ and all $m > 0$ satisfying $0 < m < b < \infty$. If we have $\limsup_{t\to\infty}(\phi(t)/t) < 1$, then there exists $b > 0$ such that $\sup_{t\in[b,\infty]}(\phi(t)/t) < 1$. For otherwise, there will be $t_n \to \infty$ with $\lim_{t\to\infty}(\phi(t_n)/t_n) = 1$, which implies that $\limsup_{t\to\infty}(\phi(t)/t) = 1$ and leads to a contradiction. Hence, $\limsup_{t\to\infty}(\phi(t)/t) < 1$ combined with the upper semicontinuity of ϕ can guarantee (2.14) for all $m > 0$ with $0 < m < b < \infty$. □

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